and the position R_0 solving Eq. (11). This process results in the following set of four equations and four variables (ω, β_0, V_0, and R_0):

$$R_0 = |\boldsymbol{R}(t_{bo}, t_a, \omega)|, \qquad V_0 = |\boldsymbol{V}(t_{bo}, t_a, \omega)|$$

$$\beta_0 = \arcsin\left[\frac{\boldsymbol{R}_{bo} \circ \boldsymbol{V}_{bo}}{(R_{bo} \cdot V_{bo})}\right]$$

$$\cos\beta_0 = \frac{R_{sat}}{2\sqrt{(\mu/R_{sat})} \cdot V_0 \cdot R_0}\left[V_0^2 + 2\mu \cdot \left(\frac{1}{R_{sat}} - \frac{1}{R_0}\right) + \frac{\mu}{R_{sat}} - \Delta V^2\right]$$

To have a closed-loop solution, ω is calculated periodically. The period was chosen as a compromise between onboard-computer load and algorithm accuracy. The value of 1 s was used in the present study. During this interval, the reference attitude was calculated by $\alpha_{3k+1} = \alpha_{3k} + \omega_k \cdot \delta t$, where δt is the onboard-computer sample interval, and the necessary frame transformation is made to get the Euler angles θ_r and ψ_r to be the reference for the attitude control system (the same transformation that is used in the pointing algorithm).

Results

To assess the performance of the guidance strategies (PA and PSG), digital simulations were done. The simulation program included most of the known nonlinearities (e.g., nonspherical Earth effects). A Monte Carlo simulation was performed, and the results were compared with the nominal trajectory without disturbances. Figure 2a shows the differences of the final orbit when no guidance was used, when only the PA algorithm was used, and when both PA and PSG algorithms were used, in the presence of a dispersion of 10% in the drag coefficient. Figure 2b shows the results of a Monte Carlo flight simulation without the PSG and the PA algorithms, which considered the random choice of the thrust profile ($3\sigma = 2\%$ of impulse) of the first and second stages, the vehicle mass ($3\sigma = 0.5\%$), the drag coefficient ($3\sigma = 10\%$), and the attitude misalignment ($3\sigma = 1$ deg) of the last stage (to orbit injection). Figure 2c shows the results of the same simulation when only the PA algorithm was included, and Fig. 2d shows the results when both the PA and the PSG algorithms were included. When both guidance strategies were used, the final orbits are so close that it is not possible to distinguish them, showing the validity of the hypotheses made.

Conclusions

The simulation results have shown that the guidance strategies PSG connected to PA fulfill an important reduction in the dispersion of the orbit parameters, resulting from errors of the first and second stages' thrust, the drag coefficient, and the pointing angle. Besides the good results in the performance, the proposed strategies have the following advantages:

1) Their algorithms are simple and easy to implement in the onboard computer.

2) The PSG algorithm can be started at any time during the burning of the third stage without any discontinuity in the attitude command (because the PSG output is the angular velocity).

3) Both strategies are suitable for solid propellant engines.

4) The PSG algorithm can set alternate altitudes if the evaluated energy is not enough to reach the nominal one.

The disadvantages detected are the following:

1) Using these strategies, it is not possible to control directly the velocity magnitude. Energy control is possible only by changing the attitude.

2) The calculation is based on an estimate of the energy available in the subsequent stages.

These results are the main reason that led the control system team to choose the PSG and the PA algorithms as the main guidance loop for the first flight of the Brazilian VLS.

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Gain-Weighted Eigenspace Assignment

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Introduction

A DIRECT eigenspace assignment method[1–4] has been used to design lateral-directional control laws for NASA's High Angle-of-Attack Research Vehicle.[5] The control laws developed have demonstrated good performance, robustness, and flying qualities during both piloted simulation and flight testing. During the control-law design effort, a limitation of the direct eigenspace assignment method became apparent—the designer has no direct control over feedback-gain magnitudes. The development of an eigenspace (eigenstructure) assignment method[6] that overcomes this limitation is presented.

Gain-Weighted Eigenspace Assignment Methodology

For a system that is observable and controllable and has n states, m controls, and l measurements, this method allows a designer to place l eigenvalues at desired locations and trade off the achievement of desired eigenvectors vs feedback-gain magnitudes. The following development assumes that complex matrices have been converted to real Jordan form. Given the observable, controllable system

$$\dot{x} = Ax + Bu \tag{1}$$

where $x \in \boldsymbol{R}^n$ and $u \in \boldsymbol{R}^m$, with system measurements available for feedback given by

$$z = Mx + Nu \tag{2}$$

where $z \in \boldsymbol{R}^l$. The total control input is the sum of the augmentation input u_c and pilot's input u_p

$$u = u_p + u_c \tag{3}$$

The measurement feedback-control law is

$$u_c = Gz \tag{4}$$

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The spectral decomposition of the closed-loop system is given by

$$\left[A + B(I_m - GN)^{-1}GM\right]V = V\Lambda \tag{5}$$

where I_m is an identity matrix of order m, V is a matrix of system eigenvectors, and Λ is a block diagonal matrix of system eigenvalues. Let W be defined by

$$W \equiv (I_m - GN)^{-1}GMV \tag{6}$$

Applying properties and rules for Kronecker products,[7] Eq. (5) can be rewritten as

$$(I_l \otimes A)\,\mathrm{vec}\,V + (I_l \otimes B)\,\mathrm{vec}\,W = \left(\Lambda^T \otimes I_n\right)\mathrm{vec}\,V \tag{7}$$

where $\otimes$ denotes Kronecker product and vec X denotes a vector formed by stacking the columns of matrix X. Solving for vec V as a function of Λ and vec W, one obtains

$$\mathrm{vec}\,V = \left[\left(\Lambda^T \otimes I_n\right) - (I_l \otimes A)\right]^{-1}(I_l \otimes B)\,\mathrm{vec}\,W \tag{8}$$

Equation (8) describes the achievable eigenvectors of the closed-loop system as a function of the desired closed-loop eigenvalues and vec W. One approach to the direct eigenspace assignment solution is to define a cost function

$$J_e = \mathrm{vec}^T(V_a - V_d)Q_d\,\mathrm{vec}(V_a - V_d) = e^T Q_d e \tag{9}$$

where $V_a =$ achievable system eigenvectors, $V_d =$ desired system eigenvectors, $Q_d =$ symmetric positive-semidefinite weighting matrix (eigenvector weighting matrix), and $e = \mathrm{vec}(V_a - V_d)$. This cost function represents the error between the achievable eigenvectors and the desired eigenvectors weighted by the matrix Q_d.

The value of vec W that will minimize J_e can be obtained by taking the partial of J_e with respect to vec W. By applying the vector chain-rule property, the partial of J_e with respect to vec W is given by

$$\frac{\partial J_e}{\partial \mathrm{vec}\,W} = 2A_D^T Q_d e = 2A_D^T Q_d(\mathrm{vec}\,V_a - \mathrm{vec}\,V_d) \tag{10}$$

In this case, a closed-form solution exists for vec W. This is obtained by setting Eq. (10) equal to zero and solving for vec W. This yields

$$\mathrm{vec}\,W = \left[A_D^T Q_d A_D\right]^{-1} A_D^T Q_d\,\mathrm{vec}\,V_d \tag{11}$$

where

$$A_D = \left[\left(\Lambda_d^T \otimes I_n\right) - (I_l \otimes A)\right]^{-1}(I_l \otimes B)$$

and Λ_d is a block-diagonal matrix of desired closed-loop eigenvalues.

The gain matrix that will place l eigenvalues to desired locations and their associated eigenvectors as close as possible to desired eigenvectors is obtained from Eq. (6).

$$G = W(MV_a + NW)^{-1} \tag{12}$$

Note in this development $(I_m - GN)$ must be nonsingular, the desired closed-loop eigenvalues cannot belong to the spectrum of A, and the unassigned eigenvalues are not guaranteed to be stable.

The gain-weighted eigenspace assignment formulation extends this direct eigenspace assignment formulation to allow trading off of eigenvector placement vs gain magnitudes, while still maintaining desired closed-loop eigenvalue locations. This is accomplished by appending a scalar measure of gain magnitude that is a function of vec W to the cost function given in Eq. (9), determining partials with respect to vec W, and solving for the optimal solution by numerical iteration.[8]

For this development, the scalar measure of gain magnitude is chosen to be a weighted sum of the squares of the individual elements of the feedback-gain matrix. A gain-magnitude cost function that allows this can be formed in terms of the vector value of G and is given by

$$J_g = \mathrm{vec}^T(G)Q_g\,\mathrm{vec}(G) = g^T Q_g g \tag{13}$$

where $Q_g =$ symmetric positive-semidefinite weighting matrix (gain-weighting matrix) and $g = \mathrm{vec}(G)$. This cost function represents the sum of the square of the individual feedback gains, weighted by the matrix Q_g. The value of vec W that yields minimum gain magnitudes while achieving the desired closed-loop eigenvalues is determined by minimizing this cost function.

Tradeoffs between achievement of desired system eigenvectors and minimizing gain magnitudes can be made by forming the composite cost function

$$J = \rho_e J_e + \rho_g J_g \tag{14}$$

where ρ_e and ρ_g are scalar positive cost-function weights on the eigenvector placement error and gain magnitudes, respectively.

Because eigenvectors can be scaled by an arbitrary constant, a unique solution to this cost function (14) does not exist when ρ_e is zero (or in practice when ρ_e is small compared to ρ_g). To ensure a unique solution for all values of ρ_e and ρ_g, it is necessary to constrain the eigenvectors to be unique. This can be accomplished by forcing one element of each eigenvector to be a specific reference value. To be consistent with the eigenvector error term J_e in Eq. (14), this specific value is chosen to be one element of each desired eigenvector. This equality constraint can be expressed in the form of a penalty function as

$$J_r = \mathrm{vec}^T(V_a - V_d)Q_r\,\mathrm{vec}(V_a - V_d) = e^T Q_r e \tag{15}$$

where $Q_r =$ symmetric positive-semidefinite weighting matrix (reference weighting matrix) and $e = \mathrm{vec}(V_a - V_d)$. This penalty function is referred to as an eigenvector reference constraint. It represents the error between an element of each achievable eigenvector and the corresponding reference element of the desired eigenvector. The weighting matrix Q_r is chosen to weight one element of each desired eigenvector.

Appending this penalty function to the cost function yields

$$J = \rho_e e^T[Q_d + (\rho_r/\rho_e)Q_r]e + \rho_g g^T Q_g g = \rho_e \bar{J}_e + \rho_g J_g \tag{16}$$

where ρ_e, ρ_g, and ρ_r are scalar positive cost-function weights on the eigenvector placement error, gain magnitudes, and the eigenvector reference constraint, respectively. With this cost function, tradeoffs between achievement of desired system eigenvectors and minimizing gain magnitudes can be made by choice of values of ρ_g and ρ_e. To ensure a unique solution, the eigenvector reference constraint weighting ρ_r should be chosen to be very large compared to the values of ρ_g and ρ_e.

The value of vec W that will minimize J can be obtained by determining the gradient of J with respect to vec W. The partial of J with respect to vec W, is given by

$$\frac{\partial J}{\partial \mathrm{vec}\,W} = \rho_e \frac{\partial \bar{J}_e}{\partial \mathrm{vec}\,W} + \rho_g \frac{\partial J_g}{\partial \mathrm{vec}\,W} \tag{17}$$

The partial of $\bar{J}_e$ with respect to vec W is given by

$$\frac{\partial \bar{J}_e}{\partial \mathrm{vec}\,W} = 2A_D^T\left(Q_d + \frac{\rho_r}{\rho_e}Q_r\right)e = 2A_D^T\left(Q_d + \frac{\rho_r}{\rho_e}Q_r\right)(\mathrm{vec}\,V_a - \mathrm{vec}\,V_d) \tag{18}$$

The partial of J_g with respect to vec W is given by

$$\frac{\partial J_g}{\partial \mathrm{vec}\,W} = 2\Big(\left[(MV_a + NW)^{-1} \otimes (I_m - GN)^T\right] - \bar{U}^T\Big\{(MV_a + NW)^{-1} \otimes \left[GM\left(\mathrm{vec}^T I_l \otimes I_n\right)(I_l \otimes A_D)\right]^T\Big\}\Big)Q_g g \tag{19}$$

where

$$\bar{U} = \left[\mathrm{vec}(I_l \otimes \mathrm{vec}\,U_{11}) \,\vdots\, \mathrm{vec}(I_l \otimes \mathrm{vec}\,U_{21}) \,\vdots\, \cdots \,\vdots\, \mathrm{vec}(I_l \otimes \mathrm{vec}\,U_{ml})\right] \tag{20}$$

and U_{ij} is a matrix of order $m \times l$, which has unity in the (i, j)th position and all other elements are zero. A complete derivation of these partials is given in Ref. 6.

Example

The design model is the lateral-directional dynamics of a high-performance aircraft at low angle of attack. The model is based on a steady-state 1-g trim flight condition of forward cruise speed equaling 598 ft/s at 25,000 ft and includes the four standard lateral-directional rigid-body degrees of freedom. The model is described by Eqs. (1–4). The system states are

$$x = [\beta \quad p_s \quad r_s \quad \phi]^T \tag{21}$$

where $\beta =$ sideslip angle (rad), $p_s =$ stability-axis roll rate (rad/s), $r_s =$ stability-axis yaw rate (rad/s), and $\phi =$ bank angle (rad). The system controls are

$$u = [a_{\text{roll}} \quad a_{\text{yaw}}]^T \tag{22}$$

where $a_{\text{roll}} =$ stability-axis roll acceleration (rad/s^2) and $a_{\text{yaw}} =$ stability-axis yaw acceleration (rad/s^2). The measurements considered for feedback are

$$z = [p_s \quad r_s \quad a_y \quad \dot{\beta}]^T \tag{23}$$

where $p_s =$ stability-axis roll rate (rad/s), $r_s =$ stability-axis yaw rate (rad/s), $a_y =$ lateral acceleration (g), and $\dot{\beta} =$ estimated sideslip rate (rad/s). The open-loop system matrices are given by

$$A = \begin{bmatrix} -0.1305 & 0.0003 & -0.9978 & 0.0537 \\ -11.199 & -1.5271 & 0.6757 & 0 \\ 2.994 & 0.1152 & -0.1529 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -0.0033 & -0.0319 \\ 1.1179 & -0.1941 \\ 0.0096 & 1.3527 \\ 0 & 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1.2576 & 0.0535 & -0.0462 & 0 \\ -0.1198 & 0.0003 & -0.9992 & 0.0538 \end{bmatrix}$$

$$N = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -0.0614 & -0.0669 \\ -0.0033 & -0.0319 \end{bmatrix}$$

The desired closed-loop eigenvalues are

$$\lambda_1 = -0.005, \qquad \lambda_2 = -2.5, \qquad \lambda_{3,4} = -1.5 \pm 1.5j$$

$$[\omega_n = 2.12 \text{ (rad/s)}, \ \zeta = 0.707]$$

The desired closed-loop eigenvectors are

$$v_1 = [0 \quad * \quad * \quad 1]^T, \qquad v_2 = [0 \quad 1 \quad * \quad *]^T$$

$$v_{3,4} = [1 \pm 0j \quad * \quad * \quad 0.0075 \pm 0.0075j]^T$$

where $j = \sqrt{-1}$ and $*$ denotes elements not weighted in the cost function.

Diagonal weighting matrices were used. Desired elements were weighted unity and other elements were weighted zero. The gain weighting matrix Q_g was set to identity. The reference weighting matrix Q_r was chosen to weight the unity elements of the desired eigenvectors. The reference constraint weighting ρ_r was set to 100.0*max(ρ_e, ρ_g).

Table 1 Example designs for values of ρ_g/ρ_e

ρ_e	ρ_g	ρ_g/ρ_e	J_e	J_g
1.0	0.0	0.0	2.0e−17	32.80
1.0	0.01	0.01	0.0077	28.17
1.0	0.05	0.05	0.1697	22.69
1.0	0.1	0.1	0.6168	16.63
1.0	0.2	0.2	1.5105	10.14
1.0	0.5	0.5	2.6575	6.152
1.0	1.0	1.0	3.3091	5.136
1.0	10.0	10.0	4.7903	4.559
0.01	1.0	100.0	5.0980	4.545
1.0e−05	1.0	1.0e+05	5.1371	4.545

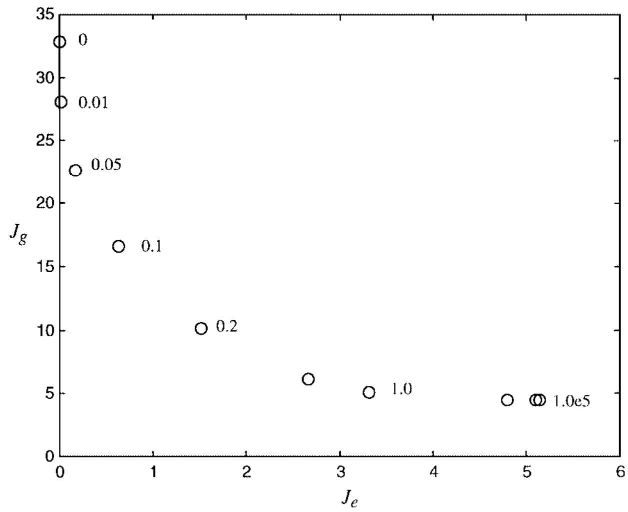

Fig. 1 Gain magnitude (J_g) vs eigenvector error (J_e) as a function of ρ_g/ρ_e; ○, ρ_g/ρ_e.

Tradeoffs between achievement of desired eigenvectors and gain magnitudes were made by varying ρ_g/ρ_e. Designs were determined for 10 values of ρ_g/ρ_e (see Table 1). A plot of J_e vs J_g is given in Fig. 1. Feedback gains for three values of ρ_g/ρ_e are as follows:

$$G = \begin{bmatrix} -0.3096 & -0.8200 & -5.1642 & -0.6697 \\ 0.0230 & 0.1138 & -0.6396 & 2.1203 \end{bmatrix}$$

$$\text{for} \quad \rho_g/\rho_e = 0.0$$

$$G = \begin{bmatrix} -0.3976 & -0.1249 & -4.489 & -0.2706 \\ 0.0830 & -1.028 & -0.6051 & 0.9296 \end{bmatrix}$$

$$\text{for} \quad \rho_g/\rho_e = 0.05$$

$$G = \begin{bmatrix} -1.0238 & 0.3554 & -0.2791 & 0.1474 \\ 0.1539 & -0.3809 & -0.9694 & 1.4704 \end{bmatrix}$$

$$\text{for} \quad \rho_g/\rho_e = 1.0e05$$

Cost-function weighting $\rho_g/\rho_e = 0.0$ yields the direct solution—four eigenvalues at desired locations and their associated eigenvectors are as close as possible in a least-squares sense to the desired eigenvectors. Cost-function weighting $\rho_g/\rho_e = 0.05$ yields four eigenvalues at desired locations and their associated eigenvectors close to the desired eigenvectors with an rms gain magnitude of 1.68, a gain magnitude reduction of 16.8% compared to the direct solution. Cost-function weighting $\rho_g/\rho_e = 1.0e05$ yields four eigenvalues at desired locations and eigenvectors that minimize feedback-gain magnitudes but have significant eigenvector error (J_e). The rms gain magnitude for this design is 0.7537; a gain magnitude reduction of approximately 63% compared to the direct solution.

Concluding Remarks

The development of the gain weighted eigenspace assignment methodology is presented. This provides a designer with a systematic methodology for trading off eigenvector placement vs gain magnitudes, while still maintaining desired closed-loop eigenvalue locations. An example demonstrated how solutions yielding achievable eigenvectors close to the desired eigenvectors could be obtained with significant reductions in gain magnitude compared to the direct eigenspace assignment solution. This result demonstrates how this method makes eigenspace assignment a much more practical and useful control-system synthesis method.

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Command Shaping in Tracking Control of a Two-Link Flexible Robot

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I. Introduction

TRACKING of a periodic trajectory by a flexible multibody sensor is important in the fields of both spacecraft dynamics and robotics. In robotics, endpoint tracking control of a single flexible arm has been demonstrated by techniques such as adaptive control,[1] input-output inversion,[2] and time-optimal control.[3,4] An alternative technique, of control augmentation with input shaping,[5] is both simple and robust and has been shown to be effective in reducing vibrations during and after slewing[6,7] and in tracking.[8] Application of multimode input shaping to tracking control of nonlinear, flexible multibody systems has not been investigated previously. This Note considers such an application, with linear or nonlinear control of a two-link flexible robot where the endpoint of the robot is to track a periodic trajectory in minimum time in a repetitive manner.

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II. Modeling

Each beam of the two-link flexible robot is modeled as a series of rigid rods connected by rotational springs, where the spring coefficients are computed by equating deflections of a cantilever beam with those of the model due to a unit tip load. This Note uses the order-n formulation of the dynamical equations of Ref. 6. Each beam is broken into rigid segments interconnected by rotational springs. The order-n equations of motion for this n degree-of-freedom system are written in terms of the $(n \times 1)$ vector q of joint and elastic rotation angles and the (2×1) joint torque vector u as

$$\ddot{q} = f(\dot{q}, q, t) + b(q)u \tag{1}$$

where the equations, not reported here for brevity, are valid for small, as well as large, deflections of the flexible links. Equation (1) is a full-order flexible body model of the two-link robot, with two rigid and four elastic degrees of freedom. For control law design the links are treated as rigid. The equations of motion for the case when both links are rigid, of identical length L and mass m driven by joint torques T_1 and T_2, are of the form

$$[M(q)]\{\ddot{q}\} + C(\dot{q}, q, t) = T \tag{2}$$

where q is a (2×1) matrix of joint angles with elements q_1 and q_2; the functional dependencies are shown in parentheses, with the meaning

$$M = \begin{bmatrix} mL^2\left(\frac{5}{3} + \cos q_2\right) & mL^2\left(\frac{1}{3} + 0.5\cos q_2\right) \\ mL^2\left(\frac{1}{3} + 0.5\cos q_2\right) & mL^2/3 \end{bmatrix} \tag{3}$$

$$C = -(0.5mL^2 \sin q_2)\begin{bmatrix} \dot{q}_2(2\dot{q}_1 + \dot{q}_2) \\ -\dot{q}_1^2 \end{bmatrix} \tag{4}$$

$$T = \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} \tag{5}$$

Now we consider independent joint control based on the reduced-order model of Eq. (2), so that measurement of beam deflections is not needed for the feedback control law.

III. Control and Inverse Kinematics

The method of feedback linearization[9] could be applied to Eq. (2), setting

$$T = M\left[\ddot{q}_c - 2\zeta\omega(\dot{q} - \dot{q}_c) - \omega^2(q - q_c)\right] + C \tag{6}$$

where M and C are given by Eqs. (3) and (4). Equation (6) ensures asymptotic decay of the tracking error. This is a model-based nonlinear control law with complex feedback. A simple linear control law can be derived from this, ignoring all coupling, with joint torques given by feedforward plus proportional-derivative feedback:

$$T_1 = \left(\tfrac{8}{3}mL^2\right)\left[\ddot{q}_{1c} - \omega^2(q_1 - q_{1c}) + 2\zeta\omega(\dot{q}_1 - \dot{q}_{1c})\right] \tag{7}$$

$$T_2 = (mL^2/3)\left[\ddot{q}_{2c} - \omega^2(q_2 - q_{2c}) + 2\zeta\omega(\dot{q}_2 - \dot{q}_{2c})\right] \tag{8}$$

where ζ and ω are the closed-loop damping and bandwidth, respectively, for controlling a rigid-body inertia. Taking ω as the first open-loop system mode, i.e., $\omega = 8.21$ Hz (note that ω is not the bandwidth of the actual flexible-body system) and $\zeta = 0.707$, and putting Eqs. (7) and (8) into Eq. (1) yield the closed-loop poles for the flexible robot as $-0.014 \pm 7.38j$, $-0.85 \pm 16j$, -37.5, -65.7, $-0.46 \pm 89.8j$, $-13.3 \pm 120.9j$, -259.6, and $-11{,}178$.

In Eqs. (7) and (8) the commanded values of the joint angles q_1 and q_2 and their first and second derivatives are obtained from the inverse kinematics of this two-link rigid-body model. Given commanded endpoint locations of the robot in inertial frame components (x_c, y_c),